

# Cops and Robber Game with a Fast Robber on Expander Graphs and Random Graphs

Abbas Mehrabian\*

University of Waterloo

## Abstract

We consider a variant of the Cops and Robber game, in which the robber has unbounded speed, i.e. can take any path from her vertex in her turn, but she is not allowed to pass through a vertex occupied by a cop. Let  $c_\infty(G)$  denote the number of cops needed to capture the robber in a graph  $G$  in this variant. We characterize graphs  $G$  with  $c_\infty(G) = 1$ , and give an  $O(|V(G)|^2)$  algorithm for their detection. We prove a lower bound for  $c_\infty$  of expander graphs, and use it to prove three things. The first is that if  $np \geq 4.2 \log n$  then the random graph  $G = \mathcal{G}(n, p)$  asymptotically almost surely has  $\eta_1/p \leq c_\infty(G) \leq \eta_2 \log(np)/p$ , for suitable constants  $\eta_1$  and  $\eta_2$ . The second is that a fixed-degree random regular graph  $G$  with  $n$  vertices asymptotically almost surely has  $c_\infty(G) = \Theta(n)$ . The third is that if  $G$  is a Cartesian product of  $m$  paths, then  $n/4km^2 \leq c_\infty(G) \leq n/k$ , where  $n = |V(G)|$  and  $k$  is the number of vertices of the longest path.

## 1 Introduction

The game of *Cops and Robber* is a perfect information game, played in a graph  $G$ . The players are a set of cops and a robber. Initially, the cops are placed at vertices of their choice in  $G$  (where more than one cop can be placed at a vertex). Then the robber, being fully aware of the cops' placement, positions herself at one of the vertices of  $G$ . Then the cops and the robber move in alternate rounds, with the cops moving first; however, players are permitted to remain stationary in their turn if they wish. The players use the

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\*amehrabi@uwaterloo.ca

edges of  $G$  to move from vertex to vertex. The cops win, and the game ends, if eventually a cop moves to the vertex currently occupied by the robber; otherwise, i.e. if the robber can elude the cops forever, the robber wins.

This game was defined (for one cop) by Winkler and Nowakowski [22] and Quilliot [24], and has been studied extensively. For a survey of results on this game, see the survey by Hahn [15]. The famous open question in this area is Meyniel’s conjecture, published by Frankl [13], which states that for every connected graph on  $n$  vertices,  $O(\sqrt{n})$  cops are sufficient to capture the robber. The best result so far is that

$$n2^{-(1-o(1))\sqrt{\log_2 n}}$$

cops are sufficient to capture the robber. This was proved independently by Lu and Peng [18], and Scott and Sudakov [25].

One interesting fact about the Cops and Robber game is that many scholars have studied the game, and yet it is not really well understood: although the upper bound  $O(\sqrt{n})$  was conjectured in 1987, no upper bound better than  $n^{1-o(1)}$  has been proved since then. As an another example, no efficient approximation algorithm for finding the number of cops needed to capture the robber in a given graph has been developed.

One might try to change the rules of the game a little in order to get a more approachable problem, and/or to understand what property of the original game causes the difficulty. Several variations of the game have been studied, by changing the rules slightly, e.g. by limiting the visibility of the cops [10], by limiting the visibility of both players [17], by changing the definition of capturing [6], or by allowing the players to move only in a certain direction along each edge [14].

The approach chosen by Fomin, Golovach, Kratochvíl, Nisse, and Suchan [12] is to allow the robber move faster than the cops. Inspired by their work, in this paper we let the robber take *any path* from her current position in her turn, but she is not allowed to pass through a vertex occupied by a cop. The parameter of interest is the *cop number* of  $G$ , which is defined as the minimum number of cops needed to ensure that the cops can win. We denote the cop number of  $G$  by  $c_\infty(G)$ , in which the  $\infty$  at the subscript indicates that the robber has unbounded speed. A nice fact about this variation is its analogy with the so-called Helicopter Cops and Robber game (defined in [26]). This is a real-time pursuit-evasion game with a robber of unbounded speed, for which Seymour and Thomas have shown that the number of cops needed equals the treewidth of the graph (which is a fairly well understood parameter) plus one [26]. However, one should not be deceived by this analogy; the cop number can be arbitrarily smaller than the treewidth:

any graph with small domination number and large treewidth (e.g., a complete graph) is such an example.

The  $c_\infty$  variant was first studied by Fomin, Golovach, Kratochvíl [11]. They proved that computing  $c_\infty(G)$  is an NP-hard problem, even if  $G$  is a split graph. (A *split graph* is a graph whose vertex set can be partitioned into a clique and an independent set.) This variant was further studied by Frieze, Krivelevich and Loh [14], who showed that for each  $n$ , there exists a connected graph on  $n$  vertices with cop number  $\Theta(n)$ . As demonstrated in [14], expansion properties of a graph are closely connected with its cop number. In this paper we further study this connection. We obtain some lower bounds for the cop number of a graph in terms of its isoperimetric numbers (Section 3). Then we use these results to give lower bounds for the cop number of random graphs (Section 4) and for the cop number of Cartesian products of graphs (Section 5).

In Section 2, we give a characterization of graphs  $G$  with  $c_\infty(G) = 1$ , and provide an  $O(|V(G)|^2)$  algorithm for deciding if  $G$  has this property. Let  $G$  be a connected graph on  $n$  vertices with maximum degree  $\Delta$ . Let  $\iota_e(G)$  and  $\iota_v(G)$  denote the edge-isoperimetric and vertex-isoperimetric numbers of  $G$ , respectively. In Section 3 we prove that for every  $G$ ,

$$c_\infty(G) \geq \max \left\{ \frac{\iota_e n}{2\Delta^2}, \frac{\iota_v n}{4\Delta} \right\}.$$

In the subsequent two sections we give some applications of this result. In Section 4 we show that if  $np \geq 4.2 \log n$ , then asymptotically almost surely the random graph  $G = \mathcal{G}(n, p)$  has

$$c_\infty(G) = \Omega\left(\frac{n}{\Delta}\right) = \Omega(1/p).$$

If also  $p = 1 - \Omega(1)$ , then we prove that asymptotically almost surely  $G$  has

$$c_\infty(G) = O(\log(np)/p).$$

In Section 4 we also show that for every fixed  $d$ , asymptotically almost surely a randomly chosen labelled  $d$ -regular graph  $G$  on  $n$  vertices has

$$c_\infty(G) = \Theta(n).$$

Let  $P_n$  and  $C_n$  denote a path and a cycle with  $n$  vertices, respectively. In Section 5 we prove that if  $G$  is the Cartesian product of  $P_{n_1}, P_{n_2}, \dots, P_{n_m}$ , where  $n_1 = \max\{n_i : 1 \leq i \leq m\}$ , then

$$\frac{n}{4n_1 m^2} \leq c_\infty(G) \leq \frac{n}{n_1}.$$

Moreover, if  $G$  is the Cartesian product of  $C_{n_1}, C_{n_2}, \dots, C_{n_m}$ , where  $n_1 = \max\{n_i : 1 \leq i \leq m\}$  is even, then

$$\frac{n}{2n_1m^2} \leq c_\infty(G) \leq \frac{2n}{n_1}.$$

In Section 6 we briefly discuss a variation in which the cops and the robber have the same speed, and we conclude with some open problems in Section 7.

## 1.1 Preliminaries and notation

Let  $G$  be the graph in which the game is played. In this paper  $G$  is always finite, and  $n$  always denote the number of vertices of  $G$ . Write  $\delta$  and  $\Delta$  for the minimum and maximum degree of  $G$ . We will assume that  $G$  is simple, because deleting multiple edges or loops does not affect the set of possible moves of the players. We consider only connected graphs, since the cop number of a disconnected graph obviously equals the sum of the cop numbers for each connected component. As we are only interested in studying the cop number, we may assume without loss of generality that the cops choose vertices of our choice in the beginning, since they can move to the vertices of their choice later.

For a subset  $A$  of vertices, the *neighbourhood* of  $A$ , written  $N(A)$ , is the set of vertices that have a neighbour in  $A$ , and the *closed neighbourhood* of  $A$ , written  $\overline{N}(A)$ , is the union  $A \cup N(A)$ . If  $A = \{v\}$  then we may write  $N(v)$  and  $\overline{N}(v)$  instead of  $N(A)$  and  $\overline{N}(A)$ , respectively. A *dominating set* is a subset  $A$  of vertices with  $V(G) = \overline{N}(A)$ , and the *domination number* of  $G$  is the minimum size of a dominating set of  $G$ . The subgraph induced by  $A$  is written  $G[A]$ , and the subgraph induced by  $V(G) - A$  is written  $G - A$ .

## 2 Characterization of Graphs with Cop Number One

For the original Cops and Robber game, graphs in which a single cop can capture the robber have been characterized independently by Nowakowski and Winkler [22] and by Quilliot [24]. In this section we characterize graphs  $G$  with  $c_\infty(G) = 1$ , and give an  $O(n^2)$  algorithm for their detection.

**Definition** (block, block tree). Let  $G$  be a connected graph. By a *block* of  $G$ , we mean either a maximal 2-connected subgraph of  $G$ , or an edge of  $G$  that is not contained in any 2-connected subgraph. We may associate with  $G$  a bipartite graph  $B(G)$  with bipartition  $(\mathcal{B}, S)$ , where  $\mathcal{B}$  is the set of blocks of  $G$  and  $S$  is the set of cut vertices of  $G$ , a block  $B$

and a cut vertex  $v$  being adjacent in  $B(G)$  if and only if  $B$  contains  $v$ . The graph  $B(G)$  is a tree, called the *block tree* of  $G$  (see for example [8], page 121).

**Lemma 2.1.** *If  $c_\infty(G) = 1$  then every block of  $G$  has domination number one.*

*Proof.* Suppose for the sake of contradiction that  $c_\infty(G) = 1$  and  $B$  is a block of  $G$  with domination number larger than one. So  $B$  is a 2-connected subgraph. Assume that there is a single cop in the game. We claim that the robber can play in such a way that, at the end of each round, if the cop is at a vertex  $v$ , then the robber is at a vertex  $r \in V(B) \setminus \overline{N}(v)$ . This shows that she can elude the cop forever, which contradicts the assumption  $c_\infty(G) = 1$ .

Assume that the cop starts at  $v_0 \in V(G)$ . Since  $B$  has domination number larger than one, there exists  $r_0 \in V(B) \setminus \overline{N}(v_0)$ . The robber starts at  $r_0$ . For every positive  $i$ , suppose that in round  $i$ , the cop moves to  $v_i$ . Since  $B$  has domination number larger than one, there exists  $r_i \in V(B) \setminus \overline{N}(v_i)$ . As  $B$  is 2-connected, there are two disjoint  $(r_{i-1}, r_i)$ -paths in  $G$ , so there exists an  $(r_{i-1}, r_i)$ -path in  $G$  that does not contain  $v_i$ . The robber has unbounded speed and moves along that path to  $r_i$ , and the proof is complete. ■

**Definition** (directed hole, hallway). Let  $u$  be a cut vertex of  $G$ , and  $B$  be a block of  $G$  containing  $u$ . If  $\{u\}$  is not a dominating set for  $B$ , then the pair  $(B, u)$  is called a *directed hole*. Let  $B, B'$  be two distinct blocks of  $G$ , and  $Bu_1 \dots u_k B'$  be the unique  $(B, B')$ -path in  $B(G)$ . If both  $(B, u_1)$  and  $(B', u_k)$  are directed holes, then the pair  $\{B, B'\}$  is called a *hallway*.

Note that if a block  $B$  is not 2-connected, then it consists of a single edge, and each of its vertices makes a dominating set. Hence, if  $\{B, B'\}$  is a hallway, then both  $B$  and  $B'$  are maximal 2-connected subgraphs. We will prove that a graph  $G$  has  $c_\infty(G) = 1$  if and only if each of its blocks has domination number one, and it does not have a hallway.

**Lemma 2.2.** *If  $c_\infty(G) = 1$ , then  $G$  does not have a hallway.*

*Proof.* Suppose for the sake of contradiction that  $c_\infty(G) = 1$  and  $\{B, B'\}$  is a hallway. By the discussion above,  $B$  and  $B'$  are maximal 2-connected subgraphs. Let  $Bu_1 \dots u_k B'$  be the unique  $(B, B')$ -path in  $B(G)$ . Assume that there is a single cop in the game. Since  $(B, u_1)$  is a directed hole, there exists  $b \in V(B) \setminus \overline{N}(u_1)$ . Similarly, since  $(B', u_k)$  is a directed hole, there exists  $b' \in V(B') \setminus \overline{N}(u_k)$ . Note that the distance between  $b$  and  $u_1$  in  $G$  is at least 2, and the distance between  $u_k$  and  $b'$  in  $G$  is at least 2, so the distance between  $b$  and  $b'$  in  $G$  is at least 4. We claim that the robber can play in such

a way that, at the end of each round, if the cop is at a vertex  $v$ , then she is at a vertex  $r \in \{b, b'\} \setminus \overline{N}(v)$ . This shows that she can elude the cop forever, which contradicts the assumption  $c_\infty(G) = 1$ .

Assume that the cop starts at  $v_0 \in V(G)$ . As the distance between  $b$  and  $b'$  in  $G$  is at least 4, there exists  $r_0 \in \{b, b'\} \setminus \overline{N}(v_0)$  and the robber starts at  $r_0$ . For every positive  $i$ , suppose that in round  $i$ , the cop moves to  $v_i$ . At the end of round  $i - 1$ , the robber was either at  $b$  or at  $b'$ , and by symmetry we may assume that she was at  $b$ . If  $b \notin \overline{N}(v_i)$ , then the robber remains at  $b$ . Otherwise  $b \in \overline{N}(v_i)$  so  $v_i \neq u$  since  $b \notin \overline{N}(u_1)$ , and  $b' \notin \overline{N}(v_i)$  since the distance between  $b$  and  $b'$  in  $G$  is at least 4. There exists two disjoint  $(b, u_1)$ -paths, thus at least one of them is cop-free. There is also a cop-free  $(u_1, u_k)$ -path and a cop-free  $(u_k, b')$ -path so the robber can move to  $b'$  in her turn. ■

The two above lemmas prove the “only if” part of the result we are going to prove. For the other direction, we need another definition and a lemma.

**Definition** (end block). Let  $G$  be a connected graph such that  $B(G)$  has more than one vertex. The blocks of  $G$  which correspond to leaves of  $B(G)$  are referred to as its *end blocks*.

**Lemma 2.3.** *Let  $B$  be an end block of graph  $G$ , and  $u$  be the unique cut vertex of  $G$  contained in  $B$ . Assume that  $\{u\}$  is a dominating set for  $B$ . Let  $H$  be the graph obtained by contracting the subgraph  $B$  into vertex  $u$ . Then we have  $c_\infty(H) \geq c_\infty(G)$ .*

*Proof.* We need to show that for every positive  $c$ , if  $c$  cops can capture the robber in  $H$ , then  $c$  cops can capture the robber in  $G$ . Assume that  $c$  cops have a capturing strategy in  $H$ . They may use the following strategy in  $G$ : whenever the robber is at a vertex  $r \in V(H)$ , they move according to their strategy in  $H$ , and when the robber moves to a vertex in  $r \in V(G) \setminus V(H)$ , they just “imagine” that the robber is at  $u$ , and again move according to their strategy in  $H$ . Since the cops’ strategy in  $H$  is winning, they eventually will either capture the robber in  $H$ , or capture the “imagined” robber at  $u$ . In the former case, the robber is captured in  $G$  as well. In the latter case, there would be a cop at  $u$  and the robber would be in  $V(G) \setminus V(H)$ . Now, that cop can capture the robber in the next move, as  $\{u\}$  is a dominating set for  $B$ , and  $V(G) \setminus V(H) \subseteq V(B)$ . ■

We are now ready to prove the main result of this section.

**Theorem 2.4.** *A connected graph  $G$  has  $c_\infty(G) = 1$  if and only if each of its blocks has domination number one, and it does not have a hallway.*

*Proof.* If  $c_\infty(G) = 1$  then by Lemma 2.1 each of the blocks of  $G$  has domination number one, and by Lemma 2.2,  $G$  does not have a hallway.

Conversely, let  $G$  be a connected graph such that each of its blocks has domination number one, and it does not have a hallway. We perform the following operation on  $G$ : let  $B$  be an arbitrary end block of  $G$ , and  $u$  be the unique cut vertex of  $G$  contained in  $B$ . If  $\{u\}$  is a dominating set for  $B$ , then we contract the subgraph  $B$  into vertex  $u$ . We repeat this operation until no such end block exists. Let  $H$  be the resulting graph. Note that each of the blocks of  $H$  is also a block of  $G$ .

*Claim.* The graph  $H$  has a single block.

*Proof of Claim.* If  $H$  has more than one block, then since  $B(H)$  is a tree, it has at least two leaves. Let  $B$  and  $B'$  be two end blocks of  $H$ ,  $u$  and  $u'$  be the unique cut vertices of  $H$  with  $u \in V(B)$  and  $u' \in V(B')$ . Since we cannot perform the above operation on  $H$ , we know that  $\{u\}$  is not a dominating set for  $B$ , and  $\{u'\}$  is not a dominating set for  $B'$ . But then  $\{B, B'\}$  would be a hallway in  $G$ , contradiction!  $\square$

Each block of  $H$  is also a block of  $G$ , hence  $H$  has domination number one, thus  $c_\infty(H) = 1$ . Lemma 2.3 gives  $c_\infty(G) \leq c_\infty(H)$ , and the proof is complete.  $\blacksquare$

We gave a mathematical characterization for graphs  $G$  with  $c_\infty(G) = 1$ . Using this we give a simple algorithm for detecting such graphs.

**Corollary 2.5.** *Let  $G$  be a connected graph on  $n$  vertices. There exists an  $O(n^2)$  algorithm to decide whether  $c_\infty(G) = 1$ .*

*Proof.* The block tree of  $G$  can be built in time  $O(|E(G)|)$  using depth-first search (see for example [8], page 142). If block  $B$  has  $m$  vertices, then it is possible to find in time  $O(m^2)$  all vertices  $u \in V(B)$  such that  $\{u\}$  is a dominating set for  $B$  (using exhaustive search). Hence in time  $O(n^2)$  one can determine if all blocks of  $G$  have domination number one, and also find all directed holes  $(B, u)$ . Using a simultaneous depth-first search on  $B(G)$  starting from all the directed holes, it is possible to decide if there is a hallway in  $G$  in time  $O(|E(B(G))|) = O(n)$ . Hence the total running time of the algorithm is  $O(n^2)$ .  $\blacksquare$

### 3 Lower Bounds for Expander Graphs

**Definition** (edge-isoperimetric number, vertex-isoperimetric number). Let  $G$  be a graph. For a subset  $S$  of vertices of  $G$ , write  $\partial S$  for the set of edges with exactly one endpoint in  $S$ . Define the *edge-isoperimetric* and *vertex-isoperimetric* numbers of  $G$  as

$$\begin{aligned}\iota_e(G) &= \min_{|S| \leq n/2} \frac{|\partial S|}{|S|}, \\ \iota_v(G) &= \min_{|S| \leq n/2} \frac{|N(S) \setminus S|}{|S|}.\end{aligned}$$

Note that for any graph  $G$  we have  $\iota_e(G) \leq \Delta$  (by taking  $S$  to be any single vertex) and  $\iota_v(G) \leq 1$  (by taking  $S$  to be any subset with  $n/2$  vertices).

In this section we prove that for every graph  $G$ , we have

$$c_\infty(G) \geq \frac{\iota_e n}{\Delta^2 - \Delta + \iota_e(\Delta + 1)} \geq \frac{\iota_e n}{2\Delta^2},$$

and

$$c_\infty(G) \geq \max \left\{ \frac{\iota_v n}{3\Delta + \iota_v(\Delta + 1)}, \frac{\iota_v n}{4\Delta} \right\}.$$

**Lemma 3.1.** *Let  $m$  be a positive integer such that for every subset  $S$  of at most  $m$  vertices,  $G - \overline{N}(S)$  has a connected component of size larger than  $n/2$ . Then  $c_\infty(G) > m$ .*

*Proof.* Assume that there are  $m$  cops in the game, and we give an escaping strategy for the robber. The strategy has the following invariant: at the end of each round, if the cops are positioned in a subset  $S$  of vertices, then the robber is at a vertex of the unique component of  $G - \overline{N}(S)$  that has size larger than  $n/2$ . Let  $S_0$  be the subset of vertices that the cops occupy when the game starts. By hypothesis,  $G - \overline{N}(S_0)$  has a connected component  $C_0$  of size larger than  $n/2$ , and the robber starts at an arbitrary vertex of  $C_0$ .

Suppose that at the end of round  $i$ , the cops are in  $S_i$ , and the robber is in a component  $C_i$  of  $G - \overline{N}(S_i)$  of size larger than  $n/2$ . In round  $i + 1$ , the cops move to a new set  $S_{i+1} \subseteq \overline{N}(S_i)$ , so the robber is not captured. Let  $C_{i+1}$  be the connected component of  $G - \overline{N}(S_{i+1})$  that has size larger than  $n/2$ . As both  $C_i$  and  $C_{i+1}$  have size larger than  $n/2$ , they intersect. Let  $v \in C_i \cap C_{i+1}$ . Since  $C_i$  is disjoint from  $\overline{N}(S_i)$ , at this moment there is no cop in  $C_i$ . Moreover,  $C_i$  is connected and the robber is in  $C_i$ , so she can move to  $v$  in this round. Hence at the end of round  $i + 1$ , the robber is in  $C_{i+1}$ , the connected component of  $G - \overline{N}(S_{i+1})$  of size larger than  $n/2$ , and the proof is complete. ■



*Remark.* The idea in the proof was first used in [14] to prove the existence of graphs with large cop number.

Before proving the main result of this section, we need a technical lemma. The proof is easy and we omit it.

**Lemma 3.2.** *Let  $n, t$  be positive integers with  $t \leq n$ . Let  $a_1, a_2, \dots, a_m$  be positive integers such that each of them is at most  $n/2$ , and their sum is  $t$ . Then we have the following.*

- (a) *One can choose a subset of  $\{a_1, \dots, a_m\}$  whose sum is between  $t/3$  and  $n/2$  (inclusive).*
- (b) *If  $t \geq n/4$  then one can choose a subset of  $\{a_1, \dots, a_m\}$  whose sum is between  $n/4$  and  $n/2$  (inclusive).*

Now we are ready to prove the main result of this section.

**Theorem 3.3.** *For every graph  $G$  we have*

- (a)  $c_\infty(G) \geq \frac{\iota_e n}{\Delta^2 - \Delta + \iota_e(\Delta + 1)} \geq \frac{\iota_e n}{2\Delta^2},$
- (b)  $c_\infty(G) \geq \frac{\iota_v n}{3\Delta + \iota_v(\Delta + 1)},$
- (c)  $c_\infty(G) \geq \frac{\iota_v n}{4\Delta}.$

*Proof.* Let  $c = c_\infty(G)$ . By Lemma 3.1 there exists a subset  $S$  of at most  $c$  vertices such that  $G - \overline{N}(S)$  has no component of size larger than  $n/2$ . We have

$$|\overline{N}(S)| \leq c(\Delta + 1), \quad |\overline{N}(S) \setminus S| \leq c\Delta, \quad \text{and} \quad |\partial\overline{N}(S)| \leq c\Delta(\Delta - 1),$$

where the last inequality holds since at most  $c\Delta$  vertices of  $\overline{N}(S)$  have a neighbour out of  $\overline{N}(S)$ , and each has at most  $\Delta - 1$  such neighbours. Let  $T = V(G) \setminus \overline{N}(S)$ , and let  $A_1, A_2, \dots, A_m$  be the connected components of  $G[T]$ . As  $G[T]$  has no component of size larger than  $n/2$ , we have  $|A_i| \leq n/2$  for all  $i$ .

(a) Since all of the  $|A_i|$ 's are at most  $n/2$ , for all  $1 \leq i \leq m$  we have  $|\partial A_i| \geq \iota_e |A_i|$ . Thus

$$|\partial T| = \sum_{i=1}^m |\partial A_i| \geq \sum_{i=1}^m \iota_e |A_i| = \iota_e \sum_{i=1}^m |A_i| = \iota_e |T|.$$

This gives

$$c\Delta(\Delta - 1) \geq |\partial\overline{N}(S)| = |\partial T| \geq \iota_e|T| = \iota_e(n - |\overline{N}(S)|) \geq \iota_e(n - c(\Delta + 1)).$$

Part (a) now results by simplifying and noting that  $\iota_e \leq \Delta$ .

(b) By Lemma 3.2 part (a), one can pick some components of  $G[T]$  such that their union,  $T'$ , has size at least  $|T|/3$  and at most  $n/2$ . Then the set  $N(T') \setminus T'$  has size at least  $\iota_v|T'|$  and at most  $|\overline{N}(S) \setminus S|$ . Thus

$$c\Delta \geq |\overline{N}(S) \setminus S| \geq \iota_v|T'| \geq \iota_v|T|/3 = \iota_v(n - |\overline{N}(S)|)/3 \geq \iota_v(n - c(\Delta + 1))/3,$$

and part (b) follows after simplification.

(c) If  $|T| < n/4$ , then we have  $|\overline{N}(S)| > 3n/4$  so that  $c(\Delta + 1) > 3n/4$  and

$$c > \frac{3n}{4(\Delta + 1)} > \frac{\iota_v n}{4\Delta},$$

as  $\iota_v \leq 1$  and  $\Delta \geq 1$ .

If  $|T| \geq n/4$ , then by Lemma 3.2 part (b), one can pick some components of  $T$  such that their union has size at least  $n/4$  and at most  $n/2$ . Let  $T'$  be their union. Then the set  $N(T') \setminus T'$  has size at least  $\iota_v|T'|$  and at most  $|\overline{N}(S) \setminus S|$ , thus

$$c\Delta \geq |\overline{N}(S) \setminus S| \geq \iota_v|T'| \geq \iota_v n/4,$$

and part (c) follows. ■

The existence of graph families with cop number  $\Theta(n)$  has been proved by Frieze et al. [14]. However, their proof is nonconstructive. A *family of bounded-degree expanders* is a sequence  $\{G_i\}_{i=1}^\infty$  of graphs, where each  $G_i$  has maximum degree  $O(1)$  and vertex-isoperimetric number  $\Omega(1)$ . Several constructions of families of bounded-degree expanders are known, see [16] for example. Thus Theorem 3.3, which shows that every family of bounded-degree expanders have cop number  $\Theta(n)$ , enables one to construct graph families with cop number  $\Theta(n)$ . This theorem also provides lower bounds for graphs with high expansion, for example random graphs (see Section 4) and Cartesian products of graphs (see Section 5).

## 4 Bounds for Random Graphs

In this section we study  $c_\infty(G)$  when  $G$  is a random graph. The original Cops and Robber game in random graphs has been studied by many authors, see for example [5, 7,

23, 19]. We denote an Erdős-Rényi random graph with parameters  $n$  and  $p$  by  $\mathcal{G}(n, p)$ . All asymptotics throughout are as  $n \rightarrow \infty$ . We say that an event in a probability space holds *asymptotically almost surely (a.a.s.)* if the probability that it holds approaches 1 as  $n$  goes to infinity. All logarithms in this section are in base  $e \approx 2.718$ . Let  $\gamma(G)$  denote the domination number of  $G$ .

The main results of this section are the following.

- Assume that  $np \geq 4.2 \log n$ . Then there exist positive constants  $\eta_1, \eta_2$  such that a random graph  $G = \mathcal{G}(n, p)$  a.a.s. has

$$\frac{\eta_1}{p} \leq c_\infty(G) \leq \frac{\eta_2 \log(np)}{p}.$$

- Assume that  $np = n^{\alpha+o(1)}$ , where  $1/2 < \alpha < 1$ . Then a.a.s.

$$c_\infty(G) = \Theta\left(\frac{\log n}{p}\right).$$

- If  $np = n^{1-o(1)}$  and  $p = 1 - \Omega(1)$ , then a.a.s.

$$c_\infty(G) = (1 + o(1)) \frac{\log n}{\log \frac{1}{1-p}}.$$

- Let  $d \geq 3$  be fixed. Then a.a.s. a randomly chosen labelled  $d$ -regular graph  $G$  on  $n$  vertices has

$$c_\infty(G) = \Theta(n).$$

We will use the following large deviation inequalities. (See Corollary A.1.10 and Theorem A.1.13 in Appendix A of [1]).

**Proposition 4.1.** *Let  $Y_1, \dots, Y_m$  be independent indicator random variables such that for all  $i$ ,  $\mathbb{E}[Y_i] = p = 1 - q$ . Let  $Y = Y_1 + \dots + Y_m$  and  $a > 0$ . Then we have the following inequalities.*

$$\Pr[Y - \mathbb{E}Y < -a] < \exp \left[ a - (a + qm) \log \left( 1 + \frac{a}{qm} \right) \right].$$

$$\Pr[Y - \mathbb{E}Y < -a] < \exp(-a^2/2pm).$$

Next we give a lower bound for vertex-isoperimetric number of random graphs, which is of independent interest. Such a bound does not seem to have appeared explicitly before.

**Theorem 4.2.** Let  $0 < b < 1$  be fixed. Let  $\beta = 1 - b$  and let  $t, k$  be constants such that

$$t > \frac{1 + \log 2}{\beta} - \log \beta, \quad k > \frac{2t}{1 - e^{-t}}.$$

If  $np \geq k \log n$  then the random graph  $G = \mathcal{G}(n, p)$  a.a.s. has  $\iota_v(G) \geq b$ .

*Proof.* We show that the random graph  $G = \mathcal{G}(n, 1 - e^{-p})$  a.a.s. has  $\iota_v(G) \geq b$ . This proves the theorem, since  $p \geq 1 - e^{-p}$  and  $\iota_v(G)$  does not decrease by adding edges to  $G$ . Let  $V(G) = \{v_1, \dots, v_n\}$ . For  $1 \leq r \leq n/2$ , define

$$A^{(r)} = \{v_{n-r+1}, \dots, v_n\}, \quad X^{(r)} = |N(A^{(r)})|.$$

Note that  $|A^{(r)}| = r$  and  $X^{(r)} = X_1^{(r)} + \dots + X_{n-r}^{(r)}$ , where  $X_i^{(r)}$  is the indicator random variable for  $v_i \in N(A^{(r)})$ . For all  $1 \leq i \leq n - r$  we have

$$\mathbb{E}X_i^{(r)} = \mathbf{Pr}[v_i \in N(A^{(r)})] = 1 - e^{-pr}.$$

By symmetry (among the subsets of size  $r$ ) and the union bound it suffices to prove that

$$\sum_{r=1}^{n/2} \binom{n}{r} \mathbf{Pr}[X^{(r)} < br] = o(1).$$

We split this sum into two parts:  $1 \leq r < t/p$  and  $t/p \leq r \leq n/2$ .

First, let  $t/p \leq r \leq n/2$ . Let  $m = n - r$ ,  $Y_i = X_i^{(r)}$  and  $a = (n - r)(1 - e^{-pr}) - br$ . The first inequality in Proposition 4.1 gives

$$\mathbf{Pr}[X^{(r)} < br] < \exp \left[ (n - r)(1 - e^{-pr}) - br - (n - r - br) \log \left( e^{pr} - \frac{bre^{pr}}{n - r} \right) \right].$$

Recall that  $\beta = 1 - b$ . Then  $1 - \frac{br}{n-r} \geq 1 - b = \beta$  so that

$$n - r - br \geq \beta(n - r), \quad e^{pr} - \frac{bre^{pr}}{n - r} \geq \beta e^{pr}$$

Thus, we have

$$\begin{aligned} \sum_{r=t/p}^{n/2} \binom{n}{r} \mathbf{Pr}[X^{(r)} < br] &\leq 2^n \exp \left[ (n - r)(1 - e^{-pr}) - br - (n - r - br)(pr + \log \beta) \right] \\ &\leq \exp \left[ n \log 2 + (n - r)(1 - \beta pr - \beta \log \beta) \right] \\ &= \left( \exp \left[ \log 2 + \frac{n - r}{n} (1 - \beta pr - \beta \log \beta) \right] \right)^n. \end{aligned}$$

To show the latter is  $o(1)$ , we need to show that  $f_1(r) = \frac{n-r}{n}(\beta pr + \beta \log \beta - 1) > \log 2$  if  $t/p \leq r \leq n/2$ . The function  $f_1(r)$  is concave on  $[t/p, n/2]$  and hence achieves its minimum at an endpoint of this interval.

If  $r = t/p$ , then

$$f_1(r) = \left(1 - \frac{t}{np}\right)(\beta t + \beta \log \beta - 1).$$

Since  $t$  was chosen so that  $\beta(t + \log \beta) - 1 > \log 2$ , and  $np = \omega(1)$ ,  $f_1(r) > \log 2$  for  $n$  large enough. If  $r = n/2$ , then

$$f_1(r) = \frac{1}{2}(\beta pn/2 + \beta \log \beta - 1) = \omega(1).$$

Now we handle the second part,  $1 \leq r < t/p$ . Let  $m = n - r$ ,  $Y_i = X_i^{(r)}$  and  $a = (n - r)(1 - e^{-pr}) - br$ . The second inequality in Proposition 4.1 gives

$$\mathbf{Pr}[X^{(r)} < br] < \exp \left[ -\frac{((n - r)(1 - e^{-pr}) - br)^2}{2(n - r)(1 - e^{-pr})} \right] < \exp \left[ br - \frac{(n - r)(1 - e^{-pr})}{2} \right].$$

For any fixed  $r$ ,  $1 \leq r < t/p$ , we have

$$\binom{n}{r} \mathbf{Pr}[X^{(r)} < br] < \exp \left[ r \log n + br - \frac{(n - r)(1 - e^{-pr})}{2} \right].$$

Therefore, to show that

$$\sum_{r=1}^{t/p} \binom{n}{r} \mathbf{Pr}[X^{(r)} < br] = o(1),$$

it is enough to show that  $\frac{(n-r)(1-e^{-pr})}{2} - r \log n - br = \Omega(n)$  if  $1 \leq r \leq t/p$ . The function  $f_2(r) = \frac{(n-r)(1-e^{-pr})}{2} - r \log n - br$  is concave, and achieves its minimum at its endpoints.

When  $r = 1$ ,

$$f_2(r) = (n - 1)(1 - e^{-p})/2 - \log n - b = \Omega(n).$$

When  $r = \frac{tn}{k \log n} \geq t/p$ ,

$$\begin{aligned} f_2(r) &= \frac{(n - r)(1 - e^{-pr})}{2} - r \log n - br \leq \frac{(n - \frac{tn}{k \log n})(1 - e^{-t})}{2} - \frac{tn}{k \log n}(b + \log n) \\ &= n \left[ \frac{(1 - \frac{t}{k \log n})(1 - e^{-t})}{2} - \frac{t}{k} - \frac{bt}{k \log n} \right], \end{aligned}$$

which is  $\Omega(n)$  as  $k$  was chosen such that

$$\frac{1 - e^{-t}}{2} - \frac{t}{k} > 0. \quad \blacksquare$$

For upper bounds, we will use some known bounds on the domination number  $\gamma(G)$  of random graphs. The following theorem has been proved in page 4 of the book by Alon and Spencer [1].

**Theorem 4.3** ([1]). *Every graph  $G$  has*

$$\gamma(G) \leq n \frac{1 + \log(\delta + 1)}{\delta + 1}.$$

**Corollary 4.4.** *If  $np > 2 \log n$  then a random graph  $G = \mathcal{G}(n, p)$  a.a.s. has*

$$\gamma(G) = O\left(\frac{n \log \delta}{\delta}\right) = O\left(\frac{\log(np)}{p}\right).$$

*Proof.* For  $np > 2 \log n$ , a.a.s.  $\delta$  is  $\Theta(np)$ . ■

The following theorem has been proved by Bonato, Prałat, and Wang [7] when  $p = o(1)$ , and by Wieland and Godbole [27] when  $p = \Omega(1)$ .

**Theorem 4.5** ([7, 27]). *If  $p = 1 - \Omega(1)$ , then a random graph  $G = \mathcal{G}(n, p)$  a.a.s. has*

$$\gamma(G) \leq (1 + o(1)) \frac{\log n}{\log \frac{1}{1-p}}.$$

For a graph  $G$ , let  $c_1(G)$  be the minimum number of cops that can capture the robber in  $G$ , in the original Cops and Robber game (in which the robber can move only to an adjacent vertex in her turn). Then we have

$$c_1(G) \leq c_\infty(G) \leq \gamma(G).$$

The lower bound is obvious. The upper bound is easy: if the cops start by occupying a dominating set, they will capture the robber in the first round.

We are ready to prove the main theorem of this section, which provides bounds for cop numbers of the random graph  $\mathcal{G}(n, p)$  for various ranges of  $p$ .

**Theorem 4.6.** *Let  $G = \mathcal{G}(n, p)$ . Then we have the following.*

(a) *If  $np \geq 4.2 \log n$ , then a.a.s.*

$$\begin{aligned} c_\infty(G) &= \Omega\left(\frac{n}{\Delta}\right) = \Omega\left(\frac{1}{p}\right), \text{ and} \\ c_\infty(G) &= O\left(\frac{n \log \delta}{\delta}\right) = O\left(\frac{\log(np)}{p}\right). \end{aligned}$$

(b) If  $np = n^{\alpha+o(1)}$  with  $\frac{1}{2} < \alpha < 1$ , then a.a.s.

$$c_\infty(G) = \Theta\left(\frac{\log n}{p}\right) = n^{1-\alpha+o(1)}.$$

(c) If  $np = n^{1-o(1)}$  and  $p = 1 - \Omega(1)$ , then a.a.s.

$$c_\infty(G) = (1 + o(1)) \frac{\log n}{\log \frac{1}{1-p}}.$$

*Proof.* (a) Let  $b = 0.001$ ,  $t = 1.7$ , and  $k = 4.2$ . It follows from Theorem 4.2 that if  $pn \geq k \log n$  then  $G$  a.a.s. has  $\iota_v(G) \geq b$ , and the lower bound follows from part (c) of Theorem 3.3, and noting that in this range we have  $\Delta = \Theta(np)$ . The upper bound follows from Corollary 4.4.

(b) Bonato, Pralat, and Wang [7] proved that if  $np = n^{\alpha+o(1)}$ , where  $1/2 < \alpha < 1$ , then a.a.s. in the original Cops and Robber game played in  $G = \mathcal{G}(n, p)$ ,

$$c_1(G) = \Theta\left(\frac{\log n}{p}\right) = n^{1-\alpha+o(1)}.$$

On the other hand, by Corollary 4.4,  $\gamma(G)$  is a.a.s at most

$$O\left(\frac{\log(np)}{p}\right) = O\left(\frac{\log n}{p}\right) = n^{1-\alpha+o(1)}.$$

The result follows since  $c_1(G) \leq c_\infty(G) \leq \gamma(G)$ .

(c) Bonato et al. [7] proved that if  $np = n^{1-o(1)}$  and  $p = 1 - \Omega(1)$ , then a.a.s. in the original Cops and Robber game played in  $G = \mathcal{G}(n, p)$ ,

$$c_1(G) = (1 + o(1)) \frac{\log n}{\log \frac{1}{1-p}}.$$

On the other hand, by Theorem 4.5,  $\gamma(G)$  is a.a.s. at most

$$(1 + o(1)) \frac{\log n}{\log \frac{1}{1-p}}.$$

The result follows since  $c_1(G) \leq c_\infty(G) \leq \gamma(G)$ .

■

Finally, we give bounds for  $c_\infty$  of random regular graphs, using the following theorem for their edge-isoperimetric number, proved by Bollobás [4].

**Theorem 4.7** ([4]). *Let  $d \geq 3$  be fixed. Then a.a.s. a randomly chosen  $d$ -regular labelled graph  $G$  on  $n$  vertices has*

$$\iota_e(G) \geq d/2 - \sqrt{d \log 2} - o(1).$$

**Corollary 4.8.** *Let  $d \geq 3$  be fixed. Then a.a.s. a randomly chosen  $d$ -regular labelled graph  $G$  on  $n$  vertices has*

$$\frac{d - 2\sqrt{d \log 2}}{4d^2} n - o(n) \leq c_\infty(G) \leq \gamma(G) \leq \frac{1 + \log(d+1)}{d+1} n.$$

*Proof.* The lower bound follows from the above bound for  $\iota_e(G)$  and part (a) of Theorem 3.3. The upper bound for  $\gamma(G)$  follows from Theorem 4.3.  $\blacksquare$

## 5 Bounds for Cartesian Products of Graphs

Let  $G_1, G_2, \dots, G_m$  be graphs. Define  $G$  to be the graph with vertex set  $V(G_1) \times V(G_2) \times \dots \times V(G_m)$  with vertices  $(u_1, u_2, \dots, u_m)$  and  $(v_1, v_2, \dots, v_m)$  being adjacent if there exists an index  $1 \leq j \leq m$  such that

- $u_i = v_i$  for all  $i \neq j$ , and
- $u_j$  and  $v_j$  are adjacent in  $G_j$ .

Then  $G$  is called the *Cartesian product* of  $G_1, G_2, \dots, G_m$ .

Neufeld and Nowakowski [21] have studied the original Cops and Robber game played on products of graphs. They have determined exactly the number of cops needed to capture the robber, when  $G$  is the Cartesian product of complete graphs, and when  $G$  is the Cartesian product of an arbitrary number of trees and cycles. In this section we study  $c_\infty(G)$  when  $G$  is a Cartesian product of graphs.

The following theorem is the main result of this section.

**Theorem 5.1.** *Let  $G_1, G_2, \dots, G_m$  be graphs and let  $n_i$  denote the number of vertices of  $G_i$  for  $1 \leq i \leq m$ . Let  $G$  be the Cartesian product of  $G_1, G_2, \dots, G_m$ , and  $n = |V(G)| = n_1 n_2 \dots n_m$ . Let  $\Delta_i$  be the maximum degree of  $G_i$ , for  $1 \leq i \leq m$ . Then we have*

$$(a) \quad \frac{\min\{\iota_e(G_i) : 1 \leq i \leq m\}n}{4(\Delta_1 + \dots + \Delta_m)^2} \leq c_\infty(G) \leq \frac{nc_\infty(G_1)}{n_1}.$$

*Note that the upper bound holds for any ordering of the graphs.*



(b) If every  $G_i$  is a path and  $n_1 = \max\{n_i : 1 \leq i \leq m\}$ , then

$$\frac{n}{4n_1m^2} \leq c_\infty(G) \leq \frac{n}{n_1}.$$

(c) If every  $G_i$  is a cycle,  $n_1 = \max\{n_i : 1 \leq i \leq m\}$ , and  $n_1$  is even, then

$$\frac{n}{2n_1m^2} \leq c_\infty(G) \leq \frac{2n}{n_1}.$$

*Remark.* When every  $G_i$  is isomorphic to an edge, it has been proved [20] using other techniques that there exist constants  $\alpha_1, \alpha_2 > 0$  such that

$$\frac{\alpha_1 n}{m\sqrt{m}} \leq c_\infty(G) \leq \frac{\alpha_2 n}{m}.$$

*Proof.* (a) Chung and Tetali [9] have proved that

$$\iota_e(G) \geq \min\{\iota_e(G_i) : 1 \leq i \leq m\}/2.$$

Noting that  $\Delta = \Delta_1 + \dots + \Delta_m$ , the lower bound follows from part (a) of Theorem 3.3.

For the upper bound we give a strategy for  $nc_\infty(G_1)/n_1$  cops to capture a robber in  $G$ . Let  $k = c_\infty(G_1)$ . We consider two games: the first one, which we call the *real game*, is a game with  $nk/n_1$  cops played in  $G$ ; and the second one, the *virtual game*, is a game in which  $k$  virtual cops are capturing a virtual robber in  $G_1$ . Given a winning strategy for the cops in the virtual game, we give a capturing strategy for the cops in the real game. We translate the moves of the cops from the virtual game to the real game, and translate the moves of the robber from the real game to the virtual game, in such a way that all the translated moves are valid, and if the robber is captured in the virtual game, then she is captured in the real game as well. By definition, there is a winning strategy for the cops in the virtual game. Hence, the real cops have a winning strategy in the real game.

For every virtual cop, we put  $n/n_1 = n_2n_3 \dots n_m$  real cops in the real game, such that if the virtual cop is in  $u_1 \in V(G_1)$ , then the real cops occupy  $\{u_1\} \times V(G_2) \times \dots \times V(G_m)$ . Also, if the real robber is at  $(v_1, \dots, v_m) \in G$ , then the virtual robber is at  $v_1 \in G_1$ . It is not hard to see that the real cops can move in such a way that these constraints hold throughout the games. Hence, once the virtual robber has been captured, the real robber has also been captured, and the proof is complete.

(b) Azizoğlu and Egecioğlu [3] have proved that

$$\iota_e(G) = \left\lfloor \frac{n_1}{2} \right\rfloor^{-1} \geq \frac{2}{n_1}.$$

As  $G$  has  $n$  vertices and maximum degree  $2m$ , the lower bound follows from part (a) of Theorem 3.3. The upper bound follows from part (a) of the present theorem, since  $G_1$  is a path and has  $c_\infty(G_1) = 1$ .

(c) Azizoglu and Egecioglu [2] have proved that

$$\iota_e(G) = \frac{4}{n_1}.$$

As  $G$  has  $n$  vertices and maximum degree  $2m$ , the lower bound follows from part (a) of Theorem 3.3. The upper bound follows from part (a) of the present theorem, since  $G_1$  is a cycle and has  $c_\infty(G_1) = 2$ . ■

## 6 The Same-Speed Variation

In the concluding remarks of [14] a variation is proposed in which all players have the same speed. In this short section we prove that the cop number of a graph in this variation equals the cop number of a related graph in the original Cops and Robber game, in which all players have speed one.

**Definition** ( $c_{a,b}(G)$ ,  $G_t$ ). Let  $a$  and  $b$  be positive integers. Let  $c_{a,b}(G)$  denote the cop number of  $G$  when the robber has speed  $a$  and the cops have speed  $b$ . That is, each cop can move along a path of length  $b$  in his turn, and the robber can move along a cop-free path of length  $a$  in her turn. Let  $t$  be a positive integer, and let  $G_t$  be the graph with vertex set  $V(G)$  with  $u, v \in V(G_t)$  being adjacent if their distance in  $G$  is at most  $t$ .

**Theorem 6.1.** *For any graph  $G$  and any positive integer  $t$  we have*

$$c_{t,t}(G) = c_{1,1}(G_t).$$

*Proof.* Consider the Cops and Robber game played in  $G_t$  with both players having speed one. Call this game the *original game*, and consider the Cops and Robber game played in  $G$  with all players having speed  $t$ , and call this game the *alternative game*. The set of possible moves for each player is almost the same in the two games, the only difference is that there could be a possible move for the robber in the original game, which is not possible in the alternative game: if the robber is at  $u$ , and  $v$  is a vertex at distance at most  $t$  from  $u$  (in  $G$ ), then she can always move from  $u$  to  $v$  in the original game, but, in the alternative game, all of the  $(u, v)$ -paths of length at most  $t$  may be blocked by a cop.

But, notice that if in some round of the original game, the robber moves from  $u$  to  $v$  in her turn, such that there is a  $(u, v)$ -path of length at most  $t$  in  $G$  with a cop standing at one of its internal vertices, then the robber will be captured in the next round. This is because that condition implies that the cop's vertex is at distance at most  $t$  from  $v$  (in  $G$ ), hence he can capture the robber in the next round. We deduce that such a move results in an immediate capture in the original game, and the robber better not do it. Apart from that kind of move, which we saw does not really give an advantage to the robber, the set of moves for the players are the same in the two games, and the equality follows. ■

## 7 Open Problems

In this section we present a few open questions and research directions on this game.

1. When  $np \geq 4.2 \log n$ , in part (a) of Theorem 4.6, we have a.a.s determined the cop number of  $\mathcal{G}(n, p)$  up to an  $O(\log(np))$  factor. Can one close this gap?
2. In part (b) of Theorem 5.1 we have determined  $c_\infty$  for the Cartesian product of  $m$  paths, up to an  $O(m^2)$  factor. Can one close the gap?
3. Fomin, Golovach, and Kratochvíl [11] proved that computing  $c_\infty(G)$  is NP-hard. Is this problem in NP? To show that this problem is in NP, one needs to prove that there is always an efficient way to describe the cops' strategy.

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